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Finite-time singularity in the vortex dynamics of a string

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We analyze the dynamics of a perfectly flexible string with a constant length and a vanishing inner friction. The local angular velocity of line elements in this seemingly simple mechanical system is shown to have many mathematical and physical properties in common with vorticity in the three-dimensional incompressible Euler equation. It is demonstrated that initially smooth vorticity fields lose their regularity within finite time in a self-similar process, and that the peak vorticity grows as $\omega_{\max}\sim(T-t)^{-1}$. [S1063-651X(99)15002-5]

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I. INTRODUCTION

The question of whether smooth solutions to the equations of fluid dynamics lose their regularity after a finite time is crucial for an understanding of phenomena as diverse as transition to turbulence [1], drop formation [2], and porous media convection [3]. In spite of considerable analytical [1,4,5] and numerical [6–8] work, however, even the fundamental problem of a finite-time singularity in the three-dimensional (3D) incompressible Euler equation is still unresolved. Under such circumstances continuous systems for which the existence of a finite-time singularity can be unambiguously extracted from the governing equations are of particular interest.

In the present work we describe a simple one-dimensional mechanical system which admits a self-similar solution describing finite-time singularity. Although our system—a frictionless string—is neither a solid nor a fluid, surprisingly many of its mathematical and physical properties resemble those of an ideal fluid described by the three-dimensional Euler equation. In particular, our model seems to represent an overlooked example of a nonfluid system whose dynamics is strongly controlled by a quantity which is analogous to vorticity in fluid dynamics. What is more, this quantity turns out to diverge with time according to the same law as is hypothesized (but not proved) for the three-dimensional Euler equation.

II. MATHEMATICAL MODEL

Consider a string with a vanishing cross section, constant length L, no inner friction, and perfect flexibility. The latter implies that the string can be bended and knotted without elastic resistance. (A long golden necklace under zero-gravity conditions would provide a reasonable experimental realization of this model on length scales larger than, say, 10 cm.) If the position $\mathbf{r}(s,t)$ and velocity $\mathbf{v}(s,t)=\partial\mathbf{r}/\partial t$ are known at some instant $t=0$ as a function of the arclength s (where $0\leqslant s\leqslant L$), the dynamical evolution of the system can be determined by solving the equations

$$\frac{\partial^2\mathbf{r}}{\partial t^2}=\frac{\partial}{\partial s}\left(\sigma\frac{\partial\mathbf{r}}{\partial s}\right) \tag{1}$$

subject to the constraint

$$\left(\frac{\partial\mathbf{r}}{\partial s}\right)^2=1 \tag{2}$$

Equation (1), which can be derived from the variational formulation of classical mechanics (see below) describes the acceleration of a particle at location s due to variations of the force $\mathbf{f}=\sigma\mathbf{e}$ acting in a tangential direction $\mathbf{e}=\partial\mathbf{r}/\partial s$ within the string. The strain $\sigma(s,t)$ with physical dimension m^2/s^2 (force per mass density) plays the same role in the string as does the pressure in fluid dynamics. It has to be determined as a part of the solution so as to satisfy the ''no-stretch condition'' [Eq. (2)] analogous to the incompressibility condition in fluid dynamics. The former expresses the fact that the length $|d\mathbf{r}|$ of each infinitesimal element of the string must remain constant during the evolution. Once the solution $[\mathbf{r}(s,t),\sigma(s,t)]$ of Eqs. (1) and (2) has been determined, the quantities

$$\boldsymbol{\omega}=\mathbf{e}\times\frac{\partial\mathbf{v}}{\partial s},\quad \boldsymbol{\kappa}=\frac{\partial\mathbf{e}}{\partial s} \tag{3}$$

can be defined, which characterize the dynamical and geometrical properties of the evolving string. It can be verified using the identity $\partial\mathbf{v}/\partial s=\partial\mathbf{e}/\partial t$ that $\boldsymbol{\omega}=\mathbf{e}\times\partial_t\mathbf{e}$ deserves to be called ''vorticity,'' since it describes the instantaneous angular velocity of a line element $d\mathbf{r}=\mathbf{e}ds$, whereas $\boldsymbol{\kappa}$ is the local curvature of the string. Equations (1) and (2) have the following properties in common with the incompressible 3D Euler equation.

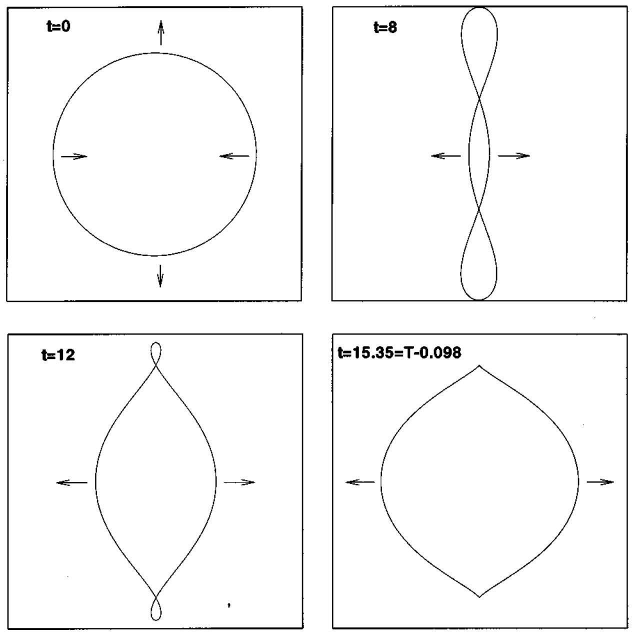

FIG. 1. Finite-time singularity in a string: Two-dimensional evolution from the initial condition $\mathbf{r}=[\cos(s)\mathbf{e}_x+\sin(s)\mathbf{e}_y]$, $\partial\mathbf{r}/\partial t=0.1[-\cos(s)\mathbf{e}_x+\sin(s)\mathbf{e}_y]$ as obtained from a numerical solution of Eqs. (1) and (2). The loops at the tips of the string at $t=15.35$ are invisibly small. All quantities are in dimensionless units.

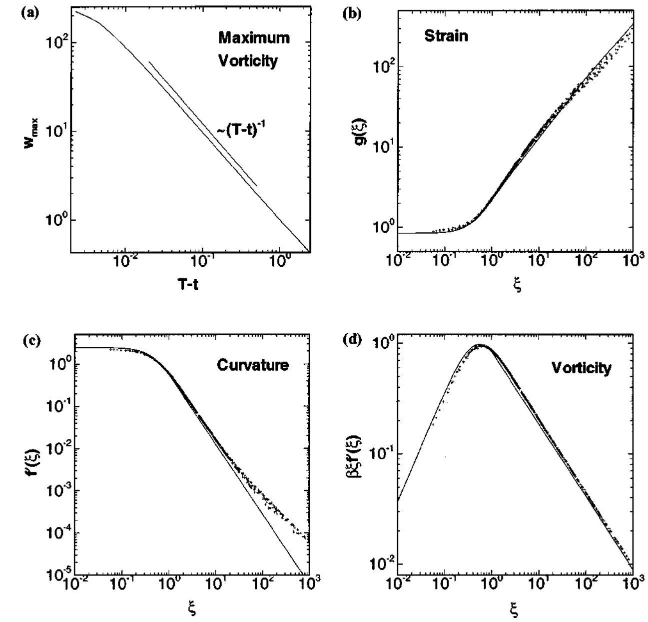

FIG. 2. Evolution of the system near the finite-time singularity: (a) Temporal behavior of the peak vorticity close to singularity time as obtained from full numerical simulation. (b)–(d) Solution as a function of the self-similar variable $\xi=s(T-t)^{-\beta}$, with $\beta=\frac{3}{2}$. Compensated strain $g(\xi)=(T-t)^{2-2\beta}\sigma$ (b), curvature $f'(\xi)=(T-t)^{\beta}\kappa$ (c), and vorticity $\beta\xi f'(\xi)=(T-t)\omega$ (d) are shown. Full lines correspond to solutions of the self-similarity equation (10) [f is obtained from Eq. (9)], and crosses correspond to time-dependent numerical simulations. All quantities are in dimensionless units.

(1) Both systems describe the motion of a continuous distribution of particles constrained solely by the requirement that the length (of the string) or the volume (of the fluid) be conserved. In the absence of the inner forces due to strain or pressure, the position of the string $\mathbf{r}(s,t)$ or the Lagrangian coordinates $\mathbf{r}(\mathbf{a},t)$ of the fluid obey $\partial^2\mathbf{r}/\partial t^2=\mathbf{0}$.

(2) Both systems are derived from variational principles with identical mathematical structures. Indeed, Eq. (1) corresponds to the extremals of the space-time integral of the kinetic energy density $(\partial\mathbf{r}/\partial t)^2$ from the initial position $\mathbf{r}(s,0)$ to the final position $\mathbf{r}(s,T)$ subject to the constraint $(\partial\mathbf{r}/\partial s)^2=1$. The Euler equation $\partial^2\mathbf{r}/\partial t^2=-\boldsymbol{\nabla}p$ corresponds to the extremals of the space-time integral of $(\partial\mathbf{r}/\partial t)^2$ from the initial Lagrangian positions $\mathbf{r}(\mathbf{a},0)=\mathbf{a}$ to the final positions $\mathbf{r}(\mathbf{a},T)$ subject to the incompressibility constraint $\mathrm{Det}(\partial\mathbf{r}/\partial\mathbf{a})=1$ [9]. The strain and pressure which have to be determined as a part of the solution appear as Lagrange multiplyers in the variational problem.

(3) Both systems are nonlocal in space due to the nonlocal character of the strain or pressure. They conserve total kinetic energy and angular momentum, while enstropy $\Omega=\int\omega^2 ds$ or, respectively, $\Omega=\int\omega^2 d\mathbf{r}$ is not in general constant.

III. NUMERICAL SOLUTION

It is well known from everyday experience that strings can be easily twisted and knotted. If there is no inner friction or elasticity to counteract the formation of increasingly small scales, the curvature and other quantities must obviously diverge within a finite time.

In Fig. 1 we show the simplest prototype of such a finite-time blowup, as obtained from a numerical solution of Eqs. (1) and (2) for the 2D motion of a closed loop using a finite-difference method with adaptive mesh refinement. In the course of evolution the initially circular string becomes distorted. After the inward-moving parts have crossed (which does not contradict the 2D dynamics if the string is infinitely thin), two loops appear whose size diminishes rapidly. We observe that both vorticity and curvature tend to infinity as the size of the small loops tends to zero. As $t\to T\approx 15.448$, the diameter of the loops shrinks to zero, the vorticity and curvature diverge, and the computational expense grows without bounds, suggesting the occurrence of a finite-time singularity. As shown in Fig. 2(a), the peak vorticity diverges as $\omega_{\max}\sim(T-t)^{-1}$, while the maximum curvature is found to behave like $\kappa_{\max}\sim(T-t)^{-3/2}$. While the curvature peaks at the tips of the small loops, say $s=0$, the vorticity is antisymmetric with respect to $s=0$. In reality, the string will become unstable against perburbations along the third dimension once the curvature becomes high enough to activate the effects of inner friction and elasticity. This effect is not considered in the present two-dimensional model.

Although the solution blows up at $t=T$, physical intuition suggests the possibility that solutions might again exist after the topological transition from a loop to a cusplike structure has occurred, i.e., for $t>T$. Indeed, if we force the numerical code to overrun the singularity time by deliberately switching off the time-step control, we obtain the result that after the ''crack of the whip'' two sharp bends emanate from the singularity point, resembling shock waves. Whereas the question of self-similar blowup in the Euler equation is an open question, we shall demonstrate below that the finite-time singularity of the string occurs in a self-similar manner.

IV. SELF-SIMILARITY SOLUTION

We start by representing $\mathbf{e}(s,t)$ for the 2D case through a single function $\phi(s,t)$ as $\mathbf{e}=\cos(\phi)\mathbf{e}_x+\sin(\phi)\mathbf{e}_y$. This choice automatically satisfies the no-stretch condition (2). After differentiation with respect to s, Eq. (1) can be rewritten in terms $\phi(s,t)$ and $\sigma(s,t)$ as

$$\phi_{tt}=\sigma\phi_{ss}+2\sigma_s\phi_s, \tag{4}$$

$$\phi_t^2=\sigma\phi_s^2-\sigma_{ss}. \tag{5}$$

In two dimensions $\boldsymbol{\omega}=\omega\mathbf{e}_z$ and $\boldsymbol{\kappa}=\kappa\mathbf{e}\times\mathbf{e}_z$ with $\omega=\phi_t$ and $\kappa=\phi_s$, respectively. We note parenthetically that if the solution of Eq. (5) is symbolically written as $\sigma=\mathcal{L}(\omega,\kappa)$, Eq. (4) can be transformed into a system of two first-order equations, namely, $\omega_t=\mathcal{L}\kappa_s+2\mathcal{L}_s\kappa$ and $\kappa_t=\omega_s$. Inspection of the first of these equations shows that intensification of local vorticity, the analog of vortex stretching in fluid dynamics, occurs as a result of either nonuniform curvature or nonuniform strain.

If self-similar behavior in the vicinity of the singularity $s=0$ and $t=T$ is expected to occur, the fields must be of the forms

$$\phi=\tau^\alpha f\left(\frac{s}{\tau^\beta}\right), \tag{6}$$

$$\sigma=\tau^\gamma g\left(\frac{s}{\tau^\beta}\right), \tag{7}$$

where we have defined $\tau=T-t$. Insertion of this ansatz into Eqs. (4) and (5) shows that $\alpha=0$ and $\gamma=2\beta-2$ are necessary conditions for self-similar behavior. Introducing the self-similarity variable $\xi=s/\tau^\beta$, the ordinary differential equations for the unknown functions $f(\xi)$ and $g(\xi)$ are readily derived from Eqs. (4) and (5) as

$$[\beta^2\xi^2-g]f_{\xi\xi}+[\beta(\beta+1)\xi-2g_\xi]f_\xi=0, \tag{8}$$

$$g_{\xi\xi}+[\beta^2\xi^2-g]f_\xi^2=0. \tag{9}$$

The quantity f_ξ can be eliminated by multiplying Eq. (8) by f_ξ and inserting f_ξ^2 from Eq. (9). This leads to a single self-similarity equation

$$(\beta^2\xi^2-g)g_{\xi\xi\xi}+(2\beta\xi-3g_\xi)g_{\xi\xi}=0. \tag{10}$$

Observe that Eq. (10) has a singular point ξ_* at $\beta^2\xi_*^2=g(\xi_*)$ (not to be confused with the point $\xi=0$ where the finite-time singularity occurs). From $f_\xi^2\geqslant 0$ and Eq. (9), it follows that $g_{\xi\xi}/(g-\beta^2\xi^2)\geqslant 0$ which demonstrates that both $g_{\xi\xi}$ and $g-\beta^2\xi^2$ must change sign across ξ_*. Thus $g_{\xi\xi}(\xi_*)=0$, i.e., the singular point is a turning point. Equation (10) is completed by the boundary condition $g_\xi(0)=0$ (from symmetry), by the matching conditions $g(\xi_*)=\beta^2\xi_*^2$, $g_\xi(\xi_*)=\beta(\beta+1)\xi_*/2$, and $g_{\xi\xi}(\xi_*)=0$, and by the condition $g_\xi\to 0$ at $\xi\to\infty$, ensuring a smooth match with the far field strain. An asymptotic solution of Eq. (10) (which for $\xi\gg 1$ simplifies to $\beta\xi g_{\xi\xi\xi}+2g_{\xi\xi}=0$) leads to the limiting behavior $g\sim\xi^{2-2/\beta}$. Since Eq. (10) is invariant under the transformations $g\to\lambda g$ and $\xi\to\lambda^{1/2}\xi$, the location of the singular point can be chosen as $\xi_*=1$ without loss of generality. As a result, there is only one free parameter $g_{\xi\xi\xi}(\xi_*)$ for the problem consisting of Eq. (10) and a total of five boundary and matching conditions. We are thus left with a nonlinear eigenvalue problem for the determination of $g(\xi)$ and the unknown scaling exponent β.

In order to solve Eq. (10), we employ a two-sided shooting method starting from ξ_*. First the free parameter $g_{\xi\xi\xi}(\xi_*)$ is determined so as to satisfy the condition $g_\xi(0)=0$. Then the solution at $\xi\to\infty$ is calculated. By repeating the computations for various values of β it turns out that the condition at infinity cannot be satisfied unless this parameter is in the vicinity of $\frac{3}{2}$. Although we do not possess a rigorous existence proof for Eq. (10) with $\beta=\frac{3}{2}$, it is likely that it represents the exact solution. The universal profiles of strain, vorticity, and curvature obtained from the self-similarity solutions are shown in Figs. 2(b)–2(d).

The existence of the self-similar solution implies that in the vicinity of the singularity

$$\sigma_{\min}\sim(T-t), \quad \omega_{\max}\sim(T-t)^{-1},$$
$$\kappa_{\max}\sim(T-t)^{-3/2}, \quad l_{\min}\sim(T-t)^{3/2}, \tag{11}$$

where $l_{\min}$ is the size of the singular region. It is interesting that the strain at the tip of the loop tends to zero as $t\to T$, which is reminiscent of low-pressure filaments in 3D turbulent shear flows [10]. Moreover, the spatial behavior of the strain far away from the singularity is characterized by $\sigma\sim s^{3/2}$, and the far field strain is independent of time.

As can be seen from Figs. 2(b)–2(d), the self-similarity solution is in excellent agreement with the behavior of the fully time-dependent simulations. In particular, the values of the compensated strain, curvature, and vorticity calculated at different times collapse onto a single curve depending on the self-similarity variable ξ. This indicates that the self-similar solution does not only exist but is also stable.

V. SUMMARY AND CONCLUSIONS

In summary, we have demonstrated that a finite-time singularity can occur in a simple one-dimensional mechanical system. It is likely that inclusion of a viscosity, i.e., $\mathbf{v}_t=(\sigma\mathbf{e})_s+\nu\mathbf{v}_{ss}$, would permit one to study questions of turbulence decay, small scale intermittency, and vorticity alignment [11]. Finally it should be noted that the present system could be experimentally studied under microgravity conditions.

A final comment is in order regarding the relation of the present results to the question of singularity in the Euler equation. It is known from the three-dimensional Euler equation [4] that the time-integral of $\omega_{\max}$ must diverge upon approach to the singularity time. If a similar theorem would hold for the present problem (which we do not know at present), our simulations would not be in contradiction to its conclusions. Introducing the quantity $l(t)=(\partial_s\omega/E)^{-2/3}$ with dimension of length, where $E=\int\mathbf{v}^2ds$ is the kinetic energy, it would follow from this theorem that $\int l(t)dt \to \infty$ is a condition for singular behavior. If blowup is self-similar with $l(t) \sim (T-t)^p$, then $p \geqslant \frac{2}{3}$. Our observation $p = \frac{3}{2}$ is not in conflict with this condition.

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[1] P. Constantin, SIAM (Soc. Ind. Appl. Math.) Rev. **36**, 73 (1992).
[2] J. Eggers, Phys. Rev. Lett. **71**, 3458 (1993); Rev. Mod. Phys. **69**, 865 (1997).
[3] R. Goldstein, A. Pesci, and M. Shelley, Phys. Rev. Lett. **70**, 3043 (1993); **75**, 3665 (1995); A. Pumir, B. I. Shraiman, and E. D. Siggia, Phys. Rev. A **45**, R5351 (1992)
[4] J. T. Beale, T. Kato, and A. Majda, Commun. Math. Phys. **94**, 61 (1984).
[5] P. Constantin, P. D. Lax, and A. Majda, Commun. Pure Appl. Math. **38**, 715 (1985).
[6] A. Pumir and E. D. Siggia, Phys. Rev. Lett. **68**, 1511 (1992); Phys. Fluids A **4**, 1472 (1993).
[7] R. Kerr, Phys. Fluids A **5**, 1725 (1993).
[8] R. Grauer, Ch. Marliani, and K. Germaschewski, Phys. Rev. Lett. **80**, 4177 (1998).
[9] J. Serrin, in *Encyclopedia of Physics, Fluid Dynamics* 1, edited by S. Fluegge and C. Truesdell (Springer-Verlag, Berlin, 1959).
[10] S. Douady, Y. Couder, and M. E. Brachet, Phys. Rev. Lett. **67**, 983 (1991); O. Cadot, S. Douady, and Y. Couder, Phys. Fluids **7**, 630 (1995).
[11] B. Galanti, J. D. Gibbon, and M. Heritage, Nonlinearity **10**, 1675 (1997).